Number Theory Practice Problems

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Review

1. 2020# 2a,c

(a) The three-digit positive integer m is odd and has three distinct digits. If the hundreds digit of m equals the product of the tens digit and ones (units) digit of m, what is m?

(b) Eleanor has 100 marbles, each of which is black or gold. The ratio of the number of black marbles to the number of gold marbles is $1:4$. How many gold marbles should she add to change this ratio to $1:6$?

(c) Suppose that n is a positive integer and that the value of $\frac{n^2+n+15}{n}$ is an integer. Determine all possible values of n.

2. 2018 #4a

(a) The positive integers 34 and 80 have exactly two positive common divisors, namely 1 and 2. How many positive integers n with $1 \leq n \leq 30$ have the property that n and 80 have exactly two positive common divisors?

3. 2017 #2a,c

(a) What are all values of x for which $x \neq 0$ and $x \neq 1$ and $\frac{5}{x(x-1)} = \frac{1}{x} + \frac{1}{x-1}$?

(b) In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. In the magic square shown, what are the values of a, b and c?

0	20	a
c	4	
	-12	b

(c) (i) For what positive integer n is $100^2 - n^2 = 9559$?

(ii) Determine one pair (a, b) of positive integers for which $a > 1$ and $b > 1$ and $ab = 9559$.

4. 2017 #5a,b

NOTE: #5b is similar to 2020 #3b

(a) Dan was born in a year between 1300 and 1400. Steve was born in a year between 1400 and 1500. Each was born on April 6 in a year that is a perfect square. Each lived for 110 years. In what year while they were both alive were their ages both perfect squares on April 7?

(b) Determine all values of k for which the points $A(1,2)$, $B(11,2)$ and $C(k,6)$ form a right-angled triangle.

5. 2016 #2a

(a) What are all values of n for which $\frac{n}{9} = \frac{25}{n}$?

6. 2016 #3b

Let n be the integer equal to $10^{20} - 20$. What is the sum of the digits of n?

7. 2016 #5b

NOTES: to be taken up with functions

8. 2016 #6a,b

(a) What is the smallest positive integer x for which $\frac{1}{32} = \frac{x}{10^y}$ for some positive integer y?

(b) Determine all possible values for the area of a right-angled triangle with one side length equal to 60 and with the property that its side lengths form an arithmetic sequence.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

9. 2008 #7

(a) The average of three consecutive multiples of 3 is a.
The average of four consecutive multiples of 4 is $a + 27$.
The average of the smallest and largest of these seven integers is 42.
Determine the value of a.

(b) Billy and Crystal each have a bag of 9 balls. The balls in each bag are numbered from 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the balls remaining in Billy's bag. Let c be the sum of the numbers on the balls remaining in Crystal's bag. Determine the probability that b and c differ by a multiple of 4.

2010 #4a

(a) Thurka bought some stuffed goats and some toy helicopters. She paid a total of \$201. She did not buy partial goats or partial helicopters. Each stuffed goat cost \$19 and each toy helicopter cost \$17. How many of each did she buy?

19g + 17h = 201

Level A problems

1. CIMC 2013 #3

If $\dfrac{99!}{101! - 99!} = \dfrac{1}{n}$, determine the value of n.

(If m is a positive integer, then $m!$ represents the product of the integers from 1 to m, inclusive. For example, $5! = 5(4)(3)(2)(1) = 120$ and $99! = 99(98)(97)\cdots(3)(2)(1)$.)

2. Euclid 2015 #1

(a) What is value of $\dfrac{10^2 - 9^2}{10 + 9}$?

(b) If $\dfrac{x+1}{x+4} = 4$, what is the value of $3x + 8$?

(c) If $f(x) = 2x - 1$, determine the value of $(f(3))^2 + 2(f(3)) + 1$.

3. Euclid 2013 #4b (move 4b to functions)

(a) How many positive integers less than 1000 have only odd digits?

(b) Determine all ordered pairs (a, b) that satisfy the following system of equations.

$$\begin{aligned} a+b &= 16 \\ \frac{4}{7} &= \frac{1}{a}+\frac{1}{b} \end{aligned}$$

4. Euclid 2014 #4

(a) Determine the number of positive divisors of 900, including 1 and 900, that are perfect squares. (A *positive divisor* of 900 is a positive integer that divides exactly into 900.)

(b) Points $A(k, 3)$, $B(3, 1)$ and $C(6, k)$ form an isosceles triangle. If $\angle ABC = \angle ACB$, determine all possible values of k.

5. Euclid 2020 #4

4.  (a) The positive integers a and b have no common divisor larger than 1. If the difference between b and a is 15 and $\frac{5}{9} < \frac{a}{b} < \frac{4}{7}$, what is the value of $\frac{a}{b}$?

 (b) A geometric sequence has first term 10 and common ratio $\frac{1}{2}$.
An arithmetic sequence has first term 10 and common difference d.
The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence.
Determine all possible values of d.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 3, 6, 12 is a geometric sequence with three terms.)

6. Euclid 2021 #4

(a) If $3(8^x) + 5(8^x) = 2^{61}$, what is the value of the real number x?

(b) For some real numbers m and n, the list $3n^2$, m^2, $2(n+1)^2$ consists of three consecutive integers written in increasing order. Determine all possible values of m.

7. AMC

In year N, the 300^{th} day of the year is a Tuesday. In year $N+1$, the 200^{th} day is also a Tuesday. On what day of the week did the 100^{th} day of year $N-1$ occur?

(A) Thursday **(B)** Friday **(C)** Saturday **(D)** Sunday **(E)** Monday

8. AMC

How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

(A) 768 **(B)** 801 **(C)** 934 **(D)** 1067 **(E)** 1167

9. AMC

Two farmers agree that pigs are worth \$300 and that goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

(A) \$5 **(B)** \$10 **(C)** \$30 **(D)** \$90 **(E)** \$210

10. AMC

What is the hundreds digit of 2011^{2011} ?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

11. CEMC Training

How many points (x, y), with x and y both integers, are on the line with equation $y = 4x + 3$ and inside the region bounded by $x = 25$, $x = 75$, $y = 120$, and $y = 250$?

(A) 44 **(B)** 36 **(C)** 40 **(D)** 32 **(E)** 48